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THE RIEMANN NON-DIFFERENTIABLE FUNCTION AND IDENTITIES FOR THE GAUSSIAN SUMS

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Riemann's example of a continuous, non-differentiable function is given by the sum $\sum_{n=1}^{\infty} \sin(n^2 x)/n^2$. This function is sufficiently irregular and its graph is fractal. Hardy proved [1] that Riemann's non-differentiable function is not differentiable at any irrational point because of a square root singularity at these points. Careful investigation of the differentiability of Riemann's non-differentiable function was carried out by Gerver, who showed [2] that this function has derivative equal to $-1/2$ at every rational point of a special type (forming the orbit of the point 1 under the theta-modular group[3]). Different proofs of this surprising fact was given by other authors[3,4,5], providing also a close relation between Riemann's non-differentiable function and classical θ-function and Gauss sums. Duistermaat obtained [3] an exact functional equations for this function under transformations of the theta-modular group. In this article we use functional equations for Riemann's non-differentiable function under theta-modular transformations to derive functional equations on Gauss sums generalizing Genocci-Schaar identity.

A Gauss sum is a sum of the form

$$S(p,q) = \sum_{n=0}^{q-1} \exp(\pi i n^2 p/q),$$

where p and q are relatively prime integers of opposite parity, i.e. one is odd and the other is even. The Genocci-Schaar identity on Gauss sums is the following: for positive integers p and q of opposite parity,

$$\frac{1}{\sqrt{q}} \sum_{n=0}^{q-1} \exp(\pi i n^2 p/q) = \frac{\exp(\pi i/4)}{\sqrt{p}} \sum_{n=0}^{p-1} \exp(-\pi i n^2 q/p).$$

This identity can be interpreted as the transformation of the Gauss sum under the change $\sigma : z \to -1/z$ where $z = p/q$.

The modular group Γ is a group of fractional linear transformations $\gamma : \ z \to (az+b)/(cz+d)$ where $a, b, c, d \in Z$ and $ad - bc = 1$. Theta-modular group Γ_θ is a sub-group of modular group generated by the following mappings: $\tau : z \to z+2$ and $\sigma : z \to -1/z$. For any element of the theta-modular group the following are valid: $ab = 0\ (mod\, 2)$ and $cd = 0\ (mod\, 2)$. Every fractional transformation $\gamma \in \Gamma_\theta$ has a simple pole at the point $z = -d/c$. The rational points $x = p/q$ with integers p, q of opposite parity and infinity constitute theta-modular orbit of point 0.

Riemann's non-differentiable function has the following local estimations at the point x: $f(x \pm \epsilon) = f(x) + R_{\pm}\epsilon^{\delta}$, where $0 < \delta \leq 1$. Using the functional equations for $f(x)$ it is possible to find functional equations on the functions $R_{\pm}$. These functions are known only at the rational points, where they coincide with the Gauss sums. The following theorems describe these functional equations:

Theorem: Let r and s be integers of which one is even and the other is odd, and s is positive, $\gamma \in \Gamma_\theta$ be an element of the theta-modular group $\gamma: \ z \to (az+b)/(cz+d)$, and suppose c is positive and $cr + ds \neq 0$. Define $r' = ar + bs$ and $s' = cr + ds$. Then the following formula for Gauss sums is valid:

$$\frac{\exp(\pi i Sign(s')/4)}{\sqrt{|s'|}} S(r', s') \frac{1}{\sqrt{c}} S(-d, c) = \frac{1}{\sqrt{s}} S(r, s) \tag{1}$$

Proof: The proof is based on the differentiability properties of Riemann's non-differentiable function. Let us consider the following function $\phi(z) = \sum_{n=1}^{\infty} \exp(\pi i n^2 z)/\pi i n^2$. Using technique of papers[4,5] we calculate the following estimation for the function $\phi(z)$ at the point $x = u/v$, where u and v are relatively prime integers of opposite parity, $(u, v) = 1$ and $uv = 0\,(mod\,2)$:

$$\phi(x + h^2) - \phi(x - h^2) = h^2 \sum_{n=-\infty}^{+\infty} \exp(\pi i n^2 x)\varphi(nh) - h^2 = \tag{2}$$

$$h^2 \sum_{t=0}^{|v|-1} \exp(\pi i t^2 x) \sum_{k=-\infty}^{+\infty} \varphi(k|v|h + th) - h^2 = 2^{1/2} S(u, v)h/|v| + O(h^2).$$

Here $\varphi(x) = \sin(\pi x^2)/\pi x^2$ if $x \neq 0$ and $\varphi(x) = 1$ if $x = 0$, and we write $n = k|v| + t$, $(0 \le t \le |v|)$ and use that $\exp(\pi i n^2 x) = \exp(\pi i t^2 x)$, since $uv = 0\,(mod\,2)$. The function $\phi(z)$ obey a functional equation under the action of theta-modular group[3]. Let $\gamma \in \Gamma_\theta$ be an element of theta modular group, then the function $\psi(z) = \phi(z) - \gamma'(z)^{-1}\mu_\gamma(z)\phi(\gamma(z))$ is differentiable, and analytical function μ_γ is given by:

$$\mu_\gamma(z) = e^{(\pi i/4)} c^{-1} (z + d/c)^{-1/2} S(-d, c). \tag{3}$$

Eq. (2) is valid for any point of theta-modular orbit of $x = r/s$, except infinity. Calculating estimation (2) for the differentiable function $\psi(x)$ at the point $x = r/s$ and supposing $\gamma(r/s) \neq \infty$, we find that

$$\frac{S(r, s)}{s} - \gamma'(r/s)^{-1/2} \mu_\gamma(r/s) \frac{S(ar + bs, cr + ds)}{|cr + ds|} = 0. \tag{4}$$

Note, that $r' = ar + bs$ and $s' = cr + ds$ are relatively prime integers. Substituting the expression of $\mu_\gamma(z)$ in terms of Gauss sums (Eq. 3) we obtain the desired result.

Q.E.D

The Genocci-Schaar identity corresponds to the case of $a = 0$, $b = -1$, $c = 1$ and $d = 0$. Functional equation (Eq. 1) describe the theta-modular transformations of Gauss sums and can be used to derive the values of the Gauss sums.

References

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